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TURBULENT CONVECTION IN A STRATIFIED BINARY MIXTURE AT A VERTICAL SURFACE

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An analytical solution is found for the nonlinear integral model of development of turbulent convection in a stratified binary mixture at a vertical surface. The character of convection can greatly differ from the case of a one-component liquid. In particular, the sign of the temperature response of the medium to heating can differ from the "intuitively expected" one, i.e., the arising ascending convective jet can be colder and not hotter than the background.

The nonlinear integral model of development of turbulent convection at a vertical surface whose temperature differs from the temperature of a liquid medium at the initial instant of time was studied earlier in [1]. In that case, the stratification of the medium was assumed to be neutral. In the present work, a more general and complex problem, i.e., convection in a binary mixture (e.g., saline water) has been considered. Here, the presence of stratification with respect to both the temperature and the concentration of an impurity is assumed. This generalization leads to qualitatively new results.

We consider a semiinfinite liquid medium in the region $x > 0$ which is bounded by a solid vertical surface $x = 0$. In this medium, constant background vertical gradients of temperature and concentration of an impurity (for brevity we will speak of salt) γ_T and γ_s, respectively, are assigned initially. The stratification of one of these components can also be unstable, but the total density stratification is assumed to be stable or neutral. The temperature of an infinite vertical surface $x = 0$ is assigned; at the instant $t = 0$ it instantly increases by a value θ_e (earlier it coincided with the temperature of the medium). Since heating of the medium from the surface does not depend on the vertical coordinate z, we will seek a solution which is independent of z. Substantiation and applicability limits of this "one-dimensional mode" of convection are discussed, e.g., in [2].

A vertically uniform convective jet, which expands with time, is formed near the heated surface. First, this jet is laminar, but as it expands and accelerates the effective Reynolds number increases, so that with not very stable density stratification the jet becomes turbulent. To describe the dynamics of the jet we will use a generalizing [1] nonlinear integral model with quadratic friction and heat exchange between the medium and the surface:

$$\frac{d(hw)}{dt} = -c\,|w|\,w + gh(\alpha\theta - \beta s)\,, \tag{1}$$

$$\frac{d(h\theta)}{dt} = c_\theta\,|w|\,(\theta_e - \theta) - \gamma_T hw\,, \tag{2}$$

$$\frac{d(hs)}{dt} = -\gamma_s hw\,, \tag{3}$$

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$$\frac{dh}{dt} = E\,|w|\ . \tag{4}$$

The Boussinesque approximation is used; the standard hypothesis for entrainment which is in proportion to the modulus of the mean velocity of the convective jet is adopted [3].*)

The first term in (1) is (with an accuracy of the constant factor) a change in the momentum of a jet element, which is determined by the effect of buoyancy and friction forces. Equation (2) describes a change in the heat content of the jet element which is determined by the nonlinear heat exchange with a heated "wall" and by vertical heat transfer (the first and second terms on the right-hand side of (2), respectively). The analog of the first of these terms is absent in the equation of impurity transfer (3), since the absence of exchange between the impurity of the medium and the wall is assumed. The sign of the modulus goes below, since it is assumed that $\theta_e > 0$ and $w \geq 0$.**)

The nonlinear system (1)–(4) allows a general solution in quadratures. If we substitute the expression for w from (4) into (2) and (3), we can consider the equations obtained as linear differential equations with respect to θ and s, respectively. Integration of them yields

$$\theta = \frac{\theta_e}{1 + E/c_\theta} - \frac{\gamma_T h}{2E + c_\theta} + \left(\theta_0 - \frac{\theta_e}{1 + E/c_\theta} + \frac{\gamma_T h_0}{2E + c_\theta}\right)\left(\frac{h_0}{h}\right)^{(1+c_\theta/E)}, \tag{5}$$

$$s = -\frac{\gamma_s h}{2E} + \left(s_0 + \frac{\gamma_s h_0}{2E}\right)\frac{h_0}{h}, \tag{6}$$

where the integration constants θ_0, s_0, and h_0 are the superheating of the convective jet, the disturbance of the concentration of the impurity (salinity), and the jet thickness at the initial instant of time $t = 0$. As the jet expands, if $(h/h_0) \xrightarrow[t \to \infty]{} \infty$, the last terms on the right-hand sides of (5) and (6) become insignificant and the solution "forgets" the initial conditions

$$\theta \approx \frac{\theta_e}{1 + E/c_\theta} - \frac{\gamma_T h}{2E + c_\theta}, \tag{7}$$

$$s \approx -\frac{\gamma_s h}{2E}\ . \tag{8}$$

In the absence of thermal stratification ($\gamma_T = 0$), expressions (5) and (7) change over to the corresponding solutions [1]. It is seen that account for stratification drastically changes the properties of solutions. In the absence of stratification, in the course of time the jet temperature tends to a constant value $\theta_e/(1 + E/c_\theta)$, which is determined by the balance of the inflow of heat from the wall and "dilution" of this heat in expansion of the jet (entrainment of the "outer" medium into the jet). In the case of a stratified medium, the last term, which increases in magnitude with increase in h, and consequently, with time, is added to (7). It describes the effect of vertical advection of heat.

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*) The assumption of the constancy of the parameter E is the most justified in the case of neutral density stratification (this case is considered in the present work). But this hypothesis is also widely used in stable stratification [3].

**) As will be seen below, in the case of a rather stable density stratification, motion can have an oscillatory character with changes of the sign of w. For this motion to be described, one must also find the solution for $w < 0$ and join this solution with the solution for $w > 0$ at the instants of rotation.

Equation (1), with account for (4) and the fact that

$$\frac{dw}{dt} = \frac{dw}{dh}\frac{dh}{dt} = Ew\frac{dw}{dh} = \frac{E}{2}\frac{d\,(w^2)}{dh}\,, \tag{9}$$

can be rewritten in the form

$$\frac{d\,(w^2)}{dh} + \frac{2}{h}\left(1 + \frac{c}{E}\right)w^2 = \frac{2g}{E}\,(\alpha\theta - \beta s)\,. \tag{10}$$

Since the dependences $\theta(h)$ and $s(h)$ are known from (5) and (6), Eq. (10) can be considered as linear with respect to w^2. Integrating this equation, we have

$$\begin{aligned} w^2 \equiv \left(\frac{1}{E}\frac{dh}{dt}\right)^2 = w_0^2\left(\frac{h_0}{h}\right)^{2(1+c/E)} + \frac{2g}{E}\Bigg\{&\left(\beta\gamma_s - \frac{\alpha\gamma_T}{1 + c_\theta/2E}\right)\left[\frac{h^2}{4E\,(2 + c/E)}\left(1 - \left(\frac{h_0}{h}\right)^{2(2+c/E)}\right)\right] + \\ &+ \frac{\alpha c_\theta \theta_e h}{E\,(1 + c_\theta/E)\,(3 + 2c/E)}\left[1 - \left(\frac{h_0}{h}\right)^{3+2c/E}\right] + \frac{\alpha h_0\left(\theta_0 - \dfrac{\theta_e}{1 + E/c_\theta} + \dfrac{\gamma_T h_0}{2E + c_\theta}\right)}{2 + 2c/E - c_\theta/E}\left(\frac{h_0}{h}\right)^{c_\theta/E} \times \\ &\times \left[1 - \left(\frac{h_0}{h}\right)^{2+2c/E-c_\theta/E}\right] - \frac{\beta h_0\left(s_0 + \dfrac{\gamma_s h_0}{2E}\right)}{2\,(1 + c/E)}\left[1 - \left(\frac{h_0}{h}\right)^{2(1+c/E)}\right]\Bigg\}\,. \end{aligned} \tag{11}$$

With respect to the function $h(t)$, equality (11) is a nonlinear differential equation of first order with separable variables, whose general solution can easily be expressed in quadratures. The problem is solved in principle.

In the general case, the solution is very cumbersome. But on rather large time intervals with $h(t) \gg h_0$ it is simplified considerably. On the right-hand side of (11), the majority of terms decrease with increase in $h(t)$, i.e., with time. If we neglect these terms (bearing the solution on rather large times in mind), the solution is expressed in elementary functions. If on the right-hand side of (11) we retain just the term ($\sim h^2$) which grows most quickly with h, we have the equation

$$\frac{dh}{dt} \approx Ih\,, \quad I = \left[\frac{g}{2\,(2 + c/E)}\left(\beta\gamma_s - \frac{\alpha\gamma_T}{1 + c_\theta/2E}\right)\right]^{1/2}. \tag{12}$$

At negative values of the expression in square brackets in (12) the solution is of an oscillatory character, which is clear since in this case we are dealing with the background state with a rather stable density stratification. This case is not considered here in detail (above we took the constraint $w(t) \geq 0$). At $I = 0$ the situation is close to that considered in [1]. When $I > 0$, the convective jet arising near the heated surface expands exponentially with time:

$$h \approx \hat{h}_0 \exp\,(It)\,. \tag{13}$$

It follows from (13), (4), (7), and (8) that

$$w \approx (\hat{h}_0 I/E) \exp(It) , \qquad (14)$$

$$\theta \approx \frac{\theta_e}{1 + E/c_\theta} - \frac{\gamma_T \hat{h}_0}{2E + c_\theta} \exp(It) , \qquad (15)$$

$$s \approx -(\gamma_s \hat{h}_0 / 2E) \exp(It) . \qquad (16)$$

The assumption used of the constancy of the coefficient of entrainment E is most justified in the absence of density stratification, i.e., when

$$\alpha\gamma_T - \beta\gamma_s = 0 . \qquad (17)$$

In this case, the increment is

$$I = \left[\frac{c_\theta \, g\beta\gamma_s}{2E\,(2 + c/E)\,(2 + c_\theta/E)} \right]^{1/2} = \left[\frac{c_\theta \, g\alpha\gamma_T}{2E\,(2 + c/E)\,(2 + c_\theta/E)} \right]^{1/2} . \qquad (18)$$

It is easily seen that in a two-component medium the properties of the solution, generally speaking, fundamentally differ from the simpler analog of the present problem [1]. First of all, exponential enhancement of the arising convective jet is possible in the case of neutral and even stable (within certain limits) density stratification. Indeed, the increment I takes on real positive values if

$$\frac{1}{\rho_0} \frac{\partial \bar{\rho}}{\partial z} > - \frac{\alpha\gamma_T}{1 + 2E/c_\theta} . \qquad (19)$$

If $\gamma_T > 0$, then, as is seen from (19), the increment can be positive even with stable (but not very strong) stratification of the density.

The expression for the temperature disturbance (15) consists of two terms. The first term is independent of time and stratification and corresponds to the steady-state solution [1] for a one-component neutral stratified medium (we recall that in the solution of [1], with power growth of h and w with time the temperature disturbance tends to a constant value). The second term in (15) increases exponentially in magnitude with time. We note that in stable temperature stratification ($\gamma_T > 0$) the second term is negative. In other words, we are dealing with a paradoxical, at first sight, result: the convective jet arising near the heated vertical surface turns out to be colder and not hotter than the environment, with the negative deviation of the temperature increasing with time. This can be interpreted in terms of "negative heat capacity," i.e., in contact with a hot surface the liquid medium is cooled rather than heated.

The physical nature of these results was analyzed in [4]. The fact is that in a stratified medium with $\gamma_T > 0$ convection leads to the ascent of a colder liquid. Therefore, the temperature of the convective jet at each level is determined not only directly by heating from the wall (the first term on the right-hand side of (15)) but also by the vertical advection of heat (the second term, which, in contrast to the first, increases with time in magnitude and is negative when $\gamma_T > 0$). At rather large times, the first term becomes small compared to the second term. The solution virtually ceases to depend on boundary and initial conditions (the situation typical of the development of instability).

Thus, the character of convection in a binary mixture near the heated vertical surface can differ qualitatively from the case of an "ordinary" one-component liquid. In particular, exponential enhancement of the arising convective turbulent jet with time is possible even in the case of stable background stratification of the density. The sign of a temperature response of the medium to heating from the vertical surface can differ from that "expected intuitively": with stable temperature stratification the ascending convective jet can be colder and not hotter than the background.

NOTATION

t, time; *z* and *x*, vertical and horizontal coordinates, respectively; θ_e, deviation of the surface temperature; $h(t)$, jet thickness; $w(t)$, $\theta(t)$, and $s(t)$, average cross-sectional vertical velocity, superheating of the jet with respect to the temperature of the medium, and disturbance of salinity, respectively; α, thermal coefficient of expansion of the medium; β, coefficient of salinity compression; *c* and c_θ, dimensionless coefficients of exchange between the surface and the medium; *E*, dimensionless coefficient of entrainment; *I*, buildup increment at the exponential stage of development of the jet; w_0, θ_0, s_0, and h_0, deviations of the corresponding quantities at the initial instant of time; $\hat{h}_0$, value of *h* at the exponential stage of development of the jet; γ_T and γ_s, background vertical gradients of temperature and concentration of the impurity; $\bar{\rho}$, background density of the medium; ρ_0 mean ("reference") value of the density; *g*, free-fall acceleration.

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